A R C H I V E S O F M E T A L L U R G Y A N D M A T E R I A L S
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ENERGY-BASED CRITERION OF ELASTIC LIMIT STATES IN FIBRE-REINFORCED COMPOSITES[1)]

ENERGETYCZNE KRYTERIUM SPRĘŻYSTYCH STANÓW GRANICZNYCH W KOMPOZYTACH WŁÓKNISTYCH

The paper deals with elastic limit states of fibre reinforced composites. In particular a single lamina of orthotropic symmetry is considered. An energy-based criterion proposed by J. Rychlewski [1, 2, 3] is specified for plane state of stress, which occurs in the considered lamina. The Rychlewski criterion was specified by calculation of the limit elastic energy densities with use of the experimental results obtained in the simple strength tests. It was shown that the condition for limit curve in biaxial state of stress, resulting from Rychlewski criterion, to be convex and bounded is equivalent to the condition that the elastic energy is positive. This conditions can be used for verification of validity of experimental parameters of elasticity and strength, which are obtained in mechanical tests applied for comosites. The known criteria of Tsai-Hill and Tsai-Wu are also identified for the same experimental data. It is visible that the differences between the discussed criteria are much larger for biaxial states of stress than for tension or compression of the lamina.

Praca zajmuje się analizą sprężystych stanów granicznych w kompozytach włóknistych. W szczególności rozważa się pojedynczą warstwę laminatu o ortotropowej symetrii sprężystej. Zastosowano energetyczne kryterium J. Rychlewskiego [1, 2, 3], które zostało wyspecjalizowane dla płaskiego stanu naprężenia występującego w laminacie. Dokonano specyfikacji tego kryterium, obliczając graniczne wartości energii sprężystej z zastosowaniem dostępnych danych doświadczalnych otrzymanych w prostych tekstach wytrzymałościowych. Wykazano, że warunek na to aby krzywa graniczna wynikająca z kryterium Rychlewskiego dla stanu dwuosiowego była wypukła i ograniczona jest równoważny żądaniu aby gęstość energii sprężystej była dodatnia. Ten warunek może być zastosowany do weryfikacji poprawności pomiaru cech sprężystych i wytrzymałościowych kompozytu. Wyspecyfikowano także

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w celach porównawczych znane kryteria Tsai-Hilla oraz Tsai-Wu. Zauważono, że różnice między badanymi kryteriami są większe dla stanów dwuosiowych w porównaniu ze stanem rozciągania lub ściskania pojedyńczej warstwy.

1. Introduction

The aim of the paper is the application of the theory of elastic eigen states based on the spectral decomposition of elastic compliance tensor **C** and elastic stiffness tensor **S** to the analysis of elastic limit states in fibre-reinforced laminar composites, cf. [1]. A single unidirectional laminae of the composite reinforced with continuous fibers is subjected to the plain state of stress, Fig. 1. The energy-based criterion of limit

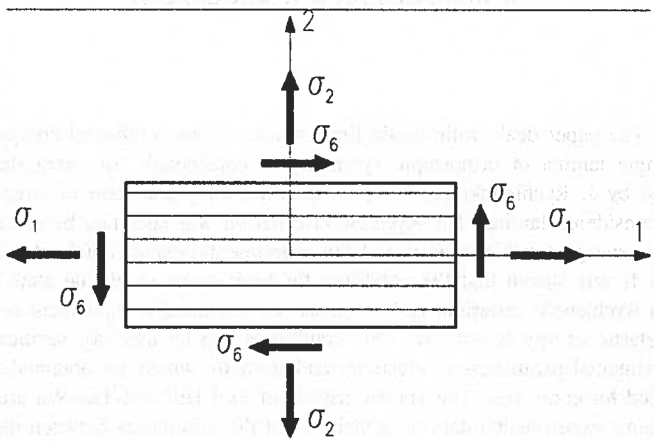

Fig. 1. A single unidirectional laminae of the composite reinforced with continuous fibres subjected to the plain state of stress

elastic states proposed by J. R y c h l e w s k i [2, 3] is applied and identified with use available experimental data. The limit elastic energy corresponding to the particular eigen state is expressed in terms the longitudinal tensile or compressive strength X_t, X_c respectively, the transverse tensile or compressive strength Y_t, Y_c and the in-plane shear strength S. According to authors knowledge, there is no reliable theory of the strength of fibre reinforced composite laminates as a whole body. Therefore there is necessary to make the strength analysis on the level of a single laminae and formulate of an algorithm of the strength assessment of the whole composite. There are two typical examples of such algorithms: the *First Ply Failure* algorithm and the *Last Ply Failure* algorithm. In both cases the crucial role plays the analysis of a single laminae. The energy-based approach provides the conditions for elastic and strength parameters of composite lamina, which guarantee that the limit curve is bounded and convex.

2. Energy-based criterion of elastic limit states

The subject of our interest is the state corresponding to the limit of linear elasticity, which corresponds to the onset of yield or fracture of the composite lamina. For such a definition of the limit state we can formulate precisely a measure of material effort as the density of elastic energy corresponding to a particular elastic eigen state, which can be determined by the symmetry of the limit tensor **H** describing the range of elastic behaviour according to the criterion for anisotropic solids of R. von Mises [4]

$$\sigma \cdot \mathbf{H} \cdot \sigma = H_{ijkl}\sigma_{ij}\sigma_{kl} \leqslant 1. \tag{1}$$

It was shown in [2, 3] that the Mises limit criterion has the energy interpretation for any anisotropic material:

$$\sigma \cdot \mathbf{H} \cdot \sigma = \frac{\Phi(\sigma_1)}{\Phi_1^e} + \cdots + \frac{\Phi\left(\sigma_p\right)}{\Phi_p^e} \leqslant 1, p \leqslant 6 \tag{2}$$
$$2\Phi(\sigma_i) = \sigma_i \cdot \mathbf{C} \cdot \sigma_i = C_{klmn}\sigma_{kl}^{(i)}\sigma_{mn}^{(i)}, i = 1, ..., p,$$

where $\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_p$ is the exactly one energy orthogonal decomposition of the stress tensor determined by the symmetry of **H**, which in our case is assumed to be the same as the symmetry of elastic compliance tensor **C**, $\Phi(\sigma_i) = \frac{\sigma_i^2}{2\lambda i}$ is the elastic energy density stored in the pertinent eigen state i, λ_i denotes the elastic Kelvin modulus in the elastic eigen state i, and Φ_i^e is the energy limit of elasticity in the elastic eigen state i, which is called the Rychlewski modulus, cf. [5], where also other papers on energy-based approach are discussed in more detail.

3. Application of Rychlewski criterion for unidirectional laminae

The unidirectional laminae can be considered as elastic body with orthotropic symmetry. Due to the negligible thickness of the laminae, the plane state of stress can be assumed. Then the elastic compliance tensor reads

$$C_{ijkl} = \begin{pmatrix} C_{1111} & C_{1122} & 0 \\ C_{1122} & C_{2222} & 0 \\ 0 & 0 & 2C_{1212} \end{pmatrix}, \tag{3}$$

where the elastic constants are expressed in the following way by the longitudinal Young modulus E_1, the transversal Young modulus E_2, and shear modulus G_{12} as well as Poisson's ratio ν_{12} of a ply

$$C_{ijkl} = \begin{pmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{2}{G_{12}} \end{pmatrix}. \tag{4}$$

The eigen problem for the elastic compliance tensor

$$\mathbf{C}\cdot\omega = \lambda\omega \tag{5}$$

leads to the condition

$$det\,(\mathbf{C} - \lambda\mathbf{1}) = 0 \tag{6}$$

which gives the eigen values called in [1] the Kelvin moduli

$$\begin{aligned}\lambda_I &= \frac{C_{1111}+C_{2222}}{2} + \frac{1}{2}\sqrt{(C_{1111}-C_{2222})^2 + 4C_{1122}^2}\\ \lambda_{II} &= \frac{C_{1111}+C_{2222}}{2} - \frac{1}{2}\sqrt{(C_{1111}-C_{2222})^2 + 4C_{1122}^2}\\ \lambda_{III} &= 2C_{1212}\end{aligned} \tag{7}$$

corresponding to the particular elastic eigen states characterized by the eigen tensors:

$$\omega_I = \begin{pmatrix} a_I & 0 & 0\\ 0 & b_I & 0\\ 0 & 0 & 0\end{pmatrix}\quad \omega_{II} = \begin{pmatrix} a_{II} & 0 & 0\\ 0 & b_{II} & 0\\ 0 & 0 & 0\end{pmatrix}\quad \omega_{III} = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{8}$$

where

$$a_I = \frac{1}{\sqrt{1+\left(\frac{\lambda_I - C_{1111}}{C_{1122}}\right)^2}}\qquad b_I = \frac{\frac{1}{C_{1122}}\left(\lambda_I - C_{1111}\right)}{\sqrt{1+\left(\frac{\lambda_I - C_{1111}}{C_{1122}}\right)^2}} \tag{9}$$

$$a_{II} = \frac{1}{\sqrt{1+\left(\frac{\lambda_{II} - C_{1111}}{C_{1122}}\right)^2}}\qquad b_{II} = \frac{\frac{1}{C_{1122}}\left(\lambda_{II} - C_{1111}\right)}{\sqrt{1+\left(\frac{\lambda_{II} - C_{1111}}{C_{1122}}\right)^2}}. \tag{10}$$

Having the Kelvin moduli the elastic energy density for a particular eigen state can be calculated for a given plane state of stress

$$\Phi_K\left(\sigma_{ij}\right) = \frac{1}{2}\lambda_K\left(\sigma_K\right)^2, \quad K = I, II, III. \tag{11}$$

Using the projectors for a particular eigen state

$$\mathbf{P}_K = \omega_K \otimes \omega_K \tag{12}$$

the energy densities can be calculated for the first eigen state:

$$\mathbf{P}_I = \omega_I \otimes \omega_I = \begin{pmatrix} a_I^2 & a_I b_I & 0\\ a_I b_I & b_I^2 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{13}$$

$$\sigma_I = \begin{pmatrix} a_I^2\sigma_1 + a_I b_I \sigma_2 \\ a_I b_I \sigma_1 + b_I^2 \sigma_2 \\ 0 \end{pmatrix} \tag{14}$$

$$\Phi_I = \frac{1}{2}\lambda_I \left[\left(a_I^2\sigma_1 + a_I b_I \sigma_2\right)^2 + \left(a_I b_I \sigma_1 + b_I^2 \sigma_2\right)^2 \right], \tag{15}$$

for the second eigen state:

$$\mathbf{P}_{II} = \omega_{II} \otimes \omega_{II} = \begin{pmatrix} a_{II}^2 & a_{II} b_{II} & 0 \\ a_{II} b_{II} & b_{II}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

$$\sigma_{II} = \begin{pmatrix} a_{II}^2\sigma_1 + a_{II} b_{II} \sigma_2 \\ a_{II} b_{II} \sigma_1 + b_{II}^2 \sigma_2 \\ 0 \end{pmatrix} \tag{17}$$

$$\Phi_{II} = \frac{1}{2}\lambda_{II} \left[\left(a_{II}^2\sigma_1 + a_{II} b_{II} \sigma_2\right)^2 + \left(a_{II} b_{II} \sigma_1 + b_{II}^2 \sigma_2\right)^2 \right] \tag{18}$$

and for the third eigen state

$$\mathbf{P}_{III} = \omega_{III} \otimes \omega_{III} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{19}$$

$$\sigma_{III} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{20}$$

$$\Phi_{III} = \frac{1}{2}\lambda_{III}\sigma_6^2. \tag{21}$$

The equations (15), (18) and (21) show how the elastic energy density is distributed over the considered elastic eigen states.

The energy-based criterion applied for the unidirectional laminae of orthotropic symmetry takes the form:

$$\frac{\Phi_I\left(\sigma_{ij}\right)}{\Phi_I^e} + \frac{\Phi_{II}\left(\sigma_{ij}\right)}{\Phi_{II}^e} + \frac{\Phi_{III}\left(\sigma_{ij}\right)}{\Phi_{III}^e} \leqslant 1. \tag{22}$$

The energy density limits of elasticity for a particular eigen state Φ_I^e, Φ_{II}^e, Φ_{III}^e are to be determined with use of experimental data obtained from the simple tests and calculated for the elastic eigen states.

4. Determination of energy limits of elasticity and specification of the energy-based criterion

Three characteristic simple tests are considered: the in-plane tension/compression along fibres, tension/compression perpendicular to the fibres and shear. These tests deliver the strength characteristics, which are assumed as elastic limits. Application of the R y c h l e w s k i criterion for each of these tests leads to the system of equations, solution of which gives the sought elastic limit energy density for a particular eigen state.

Longitudinal tension

In this case the stress tensor in the limit state is expressed according to Voigt notation in the following form:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} X \\ 0 \\ 0 \end{pmatrix} \tag{23}$$

and the corresponding elastic energy density for a particular state of stress are calculated

$$\begin{aligned} \Phi_I^L &= \tfrac{1}{2}\lambda_I X^2 \left(a_I^4 + a_I^2 b_I^2\right) \\ \Phi_{II}^L &= \frac{1}{2}\lambda_{II} X^2 \left(a_{II}^4 + a_{II}^2 b_{II}^2\right) \\ \Phi_{III}^L &= 0 \end{aligned} \tag{24}$$

and finally the Rychlewski criterion for longitudinal tension reads

$$\frac{\lambda_I X^2 \left(a_I^4 + a_I^2 b_I^2\right)}{2\Phi_I^e} + \frac{\lambda_{II} X^2 \left(a_{II}^4 + a_{II}^2 b_{II}^2\right)}{2\Phi_{II}^e} = 1. \tag{25}$$

Transversal tension

In this case the stress tensor in the limit state is expressed according to V o i g t notation in the following form:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} 0 \\ Y \\ 0 \end{pmatrix}. \tag{26}$$

The elastic energy density for a particular eigen state has the form

$$
\begin{aligned}
\Phi_I^T &= \frac{1}{2}\lambda_I Y^2\left(b_I^4 + a_I^2 b_I^2\right)\\
\Phi_{II}^T &= \frac{1}{2}\lambda_{II} Y^2\left(b_{II}^4 + a_{II}^2 b_{II}^2\right)\\
\Phi_{III}^T &= 0
\end{aligned}
\tag{27}
$$

and the Rychlewski criterion for the transversal tension reads

$$
\frac{\lambda_I Y^2\left(b_I^4 + a_I^2 b_I^2\right)}{2\Phi_I^e} + \frac{\lambda_{II} Y^2\left(b_{II}^4 + a_{II}^2 b_{II}^2\right)}{2\Phi_{II}^e} = 1. \tag{28}
$$

In-plane shear

In this case the stress tensor in the limit state is expressed according to Voigt notation in the following form:

$$
\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ S \end{pmatrix}. \tag{29}
$$

The elastic energy density for a particular eigen state has the form

$$
\begin{aligned}
{}^{S}_{I} &= 0\\
\Phi_{II}^S &= 0\\
\Phi_{III}^S &= \tfrac{1}{2}\lambda_{III} S^2
\end{aligned}
\tag{30}
$$

and the Rychlewski criterion reads

$$
\frac{\lambda_{III} S^2}{\Phi_{III}^e} = 1. \tag{31}
$$

Solving the system of equations

$$
\left.
\begin{aligned}
\frac{\Phi_I^L}{\Phi_I^e} + \frac{\Phi_{II}^L}{\Phi_{II}^e} + \frac{\Phi_{III}^L}{\Phi_{III}^e} &= 1\\
\frac{\Phi_I^T}{\Phi_I^e} + \frac{\Phi_{II}^T}{\Phi_{II}^e} + \frac{\Phi_{III}^T}{\Phi_{III}^e} &= 1\\
\frac{\Phi_I^S}{\Phi_I^e} + \frac{\Phi_{II}^S}{\Phi_{II}^e} + \frac{\Phi_{III}^S}{\Phi_{III}^e} &= 1
\end{aligned}
\right\}, \tag{32}
$$

gives the following formulae for the elastic limit energy density for a particular eigen state

$$\Phi_I^e = \lambda_I B \frac{X^2Y^2}{2\left(X^2 b_I^2 - Y^2 a_I^2\right)}$$
$$\Phi_{II}^e = \lambda_{II} B \frac{X^2Y^2}{2\left(Y^2 b_I^2 - X^2 a_I^2\right)} \tag{33}$$
$$\Phi_{III}^e = \frac{1}{2} S^2 \lambda_{III}$$
$$B = b_I^6 - a_I^6 + a_I^2 b^4 - a_I^4 b_I^2.$$

Finally the Rychlewski criterion for unidirectional laminae

$$\frac{\sigma_1^2}{X^2} + \frac{\sigma_2^2}{Y^2} - \frac{\sigma_1\sigma_2}{X^2}A + \frac{\sigma_1\sigma_2}{Y^2}A + \frac{\sigma_6^2}{S^2} \leqslant 1, \tag{34}$$

where

$$A = \frac{2a_I b_I}{b_I^2 - a_I^2} \tag{35}$$

$$a_I = \frac{1}{\sqrt{1 + \left(\frac{\lambda_I - C_{1111}}{C_{1122}}\right)^2}} \quad b_I = \frac{\frac{1}{C_{1122}}\left(\lambda_I - C_{1111}\right)}{\sqrt{1 + \left(\frac{\lambda_I - C_{1111}}{C_{1122}}\right)^2}} \tag{36}$$

$$\lambda_I = \frac{C_{1111} + C_{2222}}{2} + \frac{1}{2}\sqrt{(C_{1111} - C_{2222})^2 + 4C_{1122}^2}. \tag{37}$$

5. Discussion of the validity of the criterion

Let us discuss the consequences of the positivity of energy density

$$\Phi_K^e > 0, K = I, II, III. \tag{38}$$

For the third eigen state

$$\Phi_{III3} = \frac{1}{2}\lambda_{III} S^2 \tag{39}$$

the energy density is always positive for positive elastic moduli

$$\lambda_{III} = 2C_{1212} = 2\frac{1}{G_{12}} > 0. \tag{40}$$

The less obvious conclusions can be drawn from the condition for the first eigen state

$$\Phi_I^e = \lambda_I B \frac{X^2Y^2}{2\left(X^2 b_I^2 - Y^2 a_I^2\right)} > 0 \tag{41}$$

because due to

$$\lambda_I > 0 \tag{42}$$

$$X^2Y^2 > 0 \tag{43}$$

we have the condition for the elastic and strength parameters

$$\frac{B}{X^2b_I^2 - Y^2a_I^2} > 0. \tag{44}$$

Similarly, the analysis of the energy density for the second eigen state leads to the condition

$$\frac{B}{Y^2b_I^2 - X^2a_I^2} > 0. \tag{45}$$

The conditions (43) and (44) relate the strength parameters X and Y with the elastic constants E_1, E_2 and ν_{12} trough the coefficients B, a_I and b_I (cf. (2), $(32)_4$ and (35)). The inequalities (43) and (44) must be fulfilled simultaneously what leads to the conditions

$$B > 0 \Rightarrow X^2b_I^2 - Y^2a_I^2 > 0 \vee Y^2b_I^2 - X^2a_I^2 > 0 \tag{46}$$

$$B < 0 \Rightarrow X^2b_I^2 - Y^2a_I^2 < 0 \vee Y^2b_I^2 - X^2a_I^2 < 0. \tag{47}$$

The coefficient

$$B = b_I^6 - a_I^6 + a_I^2b^4 - a_I^4b_I^2 = \left(b_I^2 + a_I^2\right)\left(b_I^4 - a_I^4\right) \tag{48}$$

is positive iff

$$a_I < b_I \tag{49}$$

and negative iff

$$a_I > b_I. \tag{50}$$

As a result we have

$$\text{if} \quad a_I < b_I \quad \text{then} \quad \frac{X^2}{Y^2} > \frac{a_I^2}{b_I^2} \vee \frac{X^2}{Y^2} < \frac{b_I^2}{a_I^2} \tag{51}$$

$$\text{if} \quad a_I > b_I \quad \text{then} \quad \frac{X^2}{Y^2} < \frac{a_I^2}{b_I^2} \vee \frac{X^2}{Y^2} > \frac{b_I^2}{a_I^2}. \tag{52}$$

As a conclusion we can state, that the discussed criterion can be applied if the conditions (50) and (51) are fulfilled.

6. The comparison of the energy-based criterion with the commonly used strength criteria

The discussed energy-based criterion of J. Rychlewski is compared with the strength criteria known in the literature as the Tsai-Hill criterion and Tsai--Wu criterion.
The Tsai-Hill criterion takes the form [6]

$$\frac{\sigma_1^2}{X^2} + \frac{\sigma_2^2}{Y^2} - \frac{\sigma_1\sigma_2}{X^2} + \frac{\sigma_6^2}{S^2} = 1 \tag{53}$$

while the Tsai-Wu criterion reads [7]

$$F_1\sigma_1 + F_2\sigma_2 + F_6\sigma_6 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2 + 2F_{12}\sigma_1\sigma_2 = 1, \tag{54}$$

where the coefficients F_i i F_{ij} are determined in the simple uniaxial and shear tests:

$$\begin{aligned} F_{11} &= \frac{1}{X_t X_c} & F_1 &= \frac{1}{X_t} - \frac{1}{X_c} \\ F_{22} &= \frac{1}{Y_t Y_c} & F_2 &= \frac{1}{Y_1} - \frac{1}{Y_c} \\ F_{66} &= \frac{1}{S^2} & F_6 &= 0. \end{aligned} \tag{55}$$

While the coefficient F_{12} can be obtained in the biaxial test

$$F_{12} = \frac{1}{2}\sqrt{\frac{1}{\sigma^2} - \frac{F_1 + F_2}{\sigma} - (F_{11} + F_{22})} \tag{56}$$

or from the simplified formula

$$F_{12} = -\frac{\sqrt{F_{11}F_{22}}}{2}. \tag{57}$$

As an example the scheme of loading of the unidirectional lamina shown in Fig. 2 is applied. The dependence of the strength of the ply on the loading angle is studied with the application of different strength criteria. According to [8] the composite Torayca T300/Vicotex 174B (carbon/epoxy) with the following properties was used:

$$\begin{aligned} E_1 &= 148GPa, & E_2 &= 7.4GPa \\ G_{12} &= 4.8GPa, & \nu_{12} &= 0.31 \\ X_t &= 1531MPa, & X_t &= 1531MPa \\ Y_t &= 41MPa, & Y_c &= 145MPa. \\ S &= 98MPa \end{aligned}$$

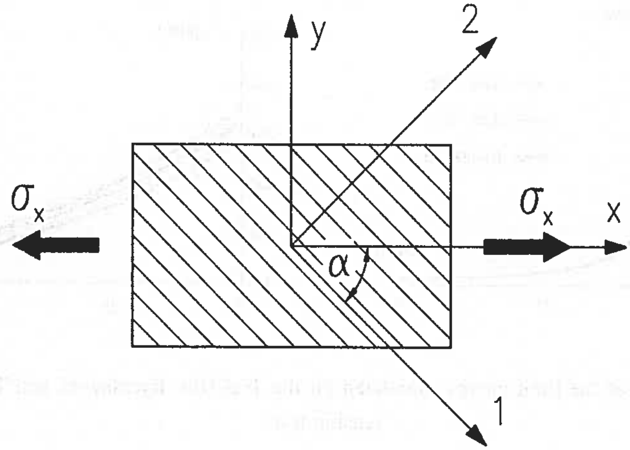

Fig. 2. Scheme of loading of the unidirectional lamina

Describing the applied stress state according to Voigt notation

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_x \cos^2 \alpha \\ \sigma_x \sin^2 \alpha \\ \sigma_x \sin \alpha \cos \alpha \end{Bmatrix} \tag{58}$$

and substituting it into the formulae of the discussed criteria the following relations for the limit stress are obtained

$$\sigma_{xATH}(\alpha) = \left[\frac{\cos^4(\alpha)}{X_t^2} + \left(\frac{1}{S^2} - \frac{1}{X_t^2} \right) \sin^2(\alpha) \cos^2(\alpha) + \frac{\sin^4(\alpha)}{Y_t^2} \right]^{-\frac{1}{2}} \tag{59}$$

$$\sigma_{xR}(\alpha) = \left[\frac{\cos^4(\alpha)}{X_t^2} + \left(\frac{1}{S^2} - \frac{A}{X_t^2} + \frac{A}{Y_t^2} \right) \sin^2(\alpha) \cos^2(\alpha) + \frac{\sin^4(\alpha)}{Y_t^2} \right]^{-\frac{1}{2}} . \tag{60}$$

In the case of T s a i - W u criterion the limit stress $\sigma_{xTW}(\alpha)$ is obtained as a solution of the equation

$$\begin{aligned} &\sigma_{xTW} \left[F_1 \cos^2(\alpha) + F_2 sin^2(\alpha) \right] + \\ &+ \sigma_{xTW}^2 \left[F_{11} \cos^4(\alpha) + F_{22} \sin^4(\alpha) + (F_{66} + 2F_{12}) \sin^2(\alpha) \cos^2(\alpha) \right] = 1, \end{aligned} \tag{61}$$

where F_i i F_{ij} are given by (54) and (56) respectively. Fig. 3 depicts the limit curves for the particular criteria discussed above. The curves are aligned very close together. Only the plots zoomed in the interval of the angle $\alpha \in (20, 30)°$ can show some differences.

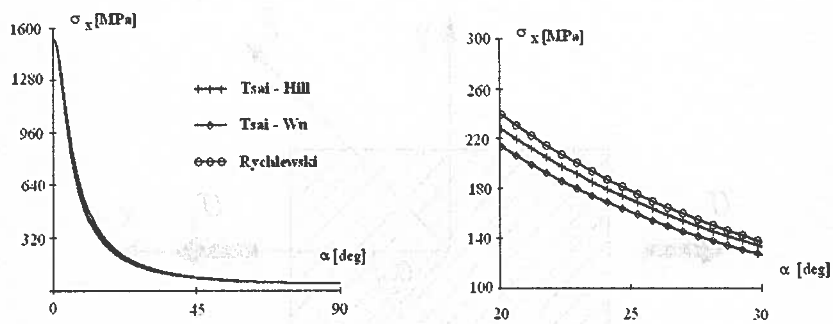

Fig. 3. Comparison of the limit curves calculated for the Tsai-Hill, Rychlewski and Tsai-Wu criteria for tension tests

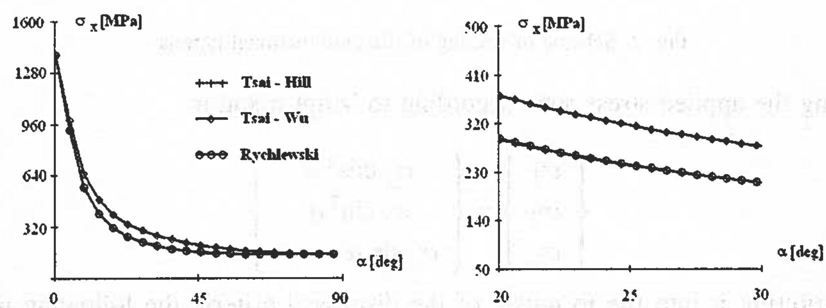

Fig. 4. Comparison of the limit curves calculated for the Tsai-Hill, Rychlewski and Tsai-Wu criteria for compression tests

More remarkable differences between plots corresponding to the Tsai-Hill, Rychlewski and Tsai-Wu criteria can be observed for compression tests given by

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} -\sigma_x \cos^2\alpha \\ -\sigma_x \sin^2\alpha \\ -\sigma_x \sin\alpha\cos\alpha \end{Bmatrix}. \tag{62}$$

In such a case in the functions (58) and (59) we put $X = X_c$ and $Y = Y_c$. The plots of limit curves are shown in Fig. 4.

7. Analysis of the limit curve for biaxial state of stress

In the case of biaxial state of stress the component $T_6 = 0$ and the Rychlewski criterion takes form

$$\frac{\sigma_1^2}{X^2} + \frac{\sigma_2^2}{Y^2} - \frac{\sigma_1\sigma_2}{X^2}A + \frac{\sigma_1\sigma_2}{Y^2}A = 1. \tag{63}$$

This formula describes for the composite lamina the limit curve, which should be convex and bounded. This requirement imposes certain conditions for elastic and strength parameters that follow from the analysis of the sign of the discriminant

$$\Delta = b^2 - ac \tag{64}$$

of the quadratic form

$$a\sigma_1^2 + 2b\sigma_1\sigma_2 + c\sigma_2^2 = 1. \tag{65}$$

If $\Delta < 0$ the equation (63) describes an ellipse, and if $\Delta > 0$ (63) corresponds to hyperbole. Then the condition $\Delta < 0$ should be assumed for the limit curve, what provides the following relations for the material parameters

$$a = \frac{1}{X^2} \quad b = \frac{1}{2}\left(\frac{A}{Y^2} - \frac{A}{X^2}\right) \quad c = \frac{1}{Y^2} \tag{66}$$

$$\left(\frac{A}{Y^2} - \frac{A}{X^2}\right)^2 - \frac{4}{X^2Y^2} < 0. \tag{67}$$

It can be shown that the condition (65) is equivalent to the conditions (50) and (51). After small rearrangements we have

$$\frac{X^2}{Y^2} + \frac{Y^2}{X^2} < \frac{b_I^2}{a_I^2} + \frac{a_I^2}{b_I^2} \tag{68}$$

and due to

$$\frac{X^2}{Y^2} = c \quad \frac{b_I^2}{a_I^2} = d \tag{69}$$

the inequality holds

$$c^2 - \left(d + \frac{1}{d}\right)c + 1 < 0. \tag{70}$$

The roots of the equation

$$c^2 - \left(d + \frac{1}{d}\right)c + 1 = 0 \tag{71}$$

are given by

$$c_1 = \frac{1}{d} \quad \text{i} \quad c_2 = d. \tag{72}$$

If

$$\frac{1}{d} < d \tag{73}$$

the solution of the inequality (68) reads

$$c > \frac{1}{d} \vee c < d \tag{74}$$

or due to (67)

$$\frac{X^2}{Y^2} > \frac{a_I^2}{b_I^2} \vee \frac{X^2}{Y^2} < \frac{b_I^2}{a_I^2}. \tag{75}$$

If on the other hand

$$\frac{1}{d} > d \tag{76}$$

then the solution of (68) si given by

$$c > d \vee c < \frac{1}{d}. \tag{77}$$

or due to (67)

$$\frac{X^2}{Y^2} < \frac{a_I^2}{b_I^2} \vee \frac{X^2}{Y^2} > \frac{b_I^2}{a_I^2} \tag{78}$$

It follows from

$$\frac{1}{d} < d \Leftrightarrow a_I < b_I \tag{79}$$

$$\frac{1}{d} > d \Leftrightarrow a_I > b_I \tag{80}$$

that the conditions (73) i (74) are equivalent to (50) i (51).
In Fig. 5. the limit curves calculated in biaxial states of stress for the discussed criteria of Rychlewski, Tsai-Hill and Tsai-Wu are shown. Due to the different values of strength in tension and compression, the Rychlewski and Tsai-Wu criteria are calculated piecewise.

It is visible that the differences between the discussed criteria are much larger for biaxial states of stress than for tension or compression of the lamina, cf. Fig. 3 and Fig. 4. Depending on the relation between the longitudinal and compression strength the both criteria can reveal non-convexity in the shape of limit curve or not. We can observe that the energy-based criterion of Rychlewski can provide more adequate results for composites, in which the strength parameters in tension X_t, Y_t and compression X_c, Y_c have respectively similar values. The experimental data obtained in biaxial tests are necessary to verify, which of the criterion is most reliable.

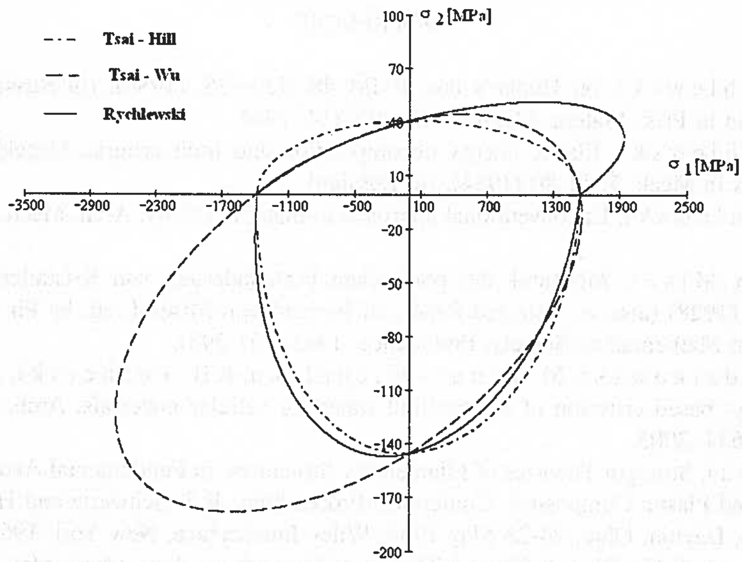

Fig. 5. Comparison of the limit curves calculated for the Tsai-Hill, Rychlewski and Tsai-Wu criteria for biaxial states of stress

8. Conclusions

Studying deformation of a single ply of fiber reinforced composite three elastic eigen states were determined and the formulae for elastic energy density accumulated in a particular eigen states were derived. The energy-based criterion of Rychlewski was specified by calculation of the limit elastic energy densities with use of the experimental results obtained in the simple strength tests. It was shown that the condition for limit curve, resulting from Rychlewski criterion, to be convex and bounded is equivalent to the condition that the elastic energy is positive (50), (51). This conditions can be used for verification of validity of experimental parameters of elasticity and strength, which are obtained in mechanical tests applied for composites. The data available in the literature show that the discrepancies in values run over 30% for the same material [8]. Therefore application of the positive energy condition (50), (51) could appear helpful in elaboration of reliable set of experimental data.

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